Eigen Values of Interval Matrix

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In this paper, properties of interval matrix are studied. Some theoretical results on the regularity and the singularity of an interval matrix are explored and verified through several examples. Eigen value of the interval matrix using singularity property is studied. Matlab code is developed to determine eigen values of an interval matrix.

Keywords : Interval Matrix, Regular Interval Matrix, Singular Interval Matrix, Eigen Value of an Interval Matrix.

1 Introduction

Interval Analysis is a tool in numerical computing where the rules for the arithmetic of intervals are explicitly stated and applied to what is called today interval arithmetic evaluation of rational expressions. Moore [3] and P.S. Dwyer [1] have discussed matrix computations using interval arithmetic in their book in 1951. The Japanese scientist Teruo Sunaga's [2] paper was one of the most important paper for the development of interval arithmetic. Jiri Rohn's paper like [5], [8], [4] etc are important in this field.

In this chapter we have discussed some priliminary on interval matrix operations.

1.1 Notations

X = interval
$\underline{X}$ and $\overline{X}$ = left endpoint and right endpoint of an interval respectively
A^I = Interval matrix
A_c = Centre of the interval matrix
Δ = Spread of the interval matrix
P matrix = Whose principal minors are greater than 0
Δ_1 and Δ_2 = The first and the second principal minor respectively
T_y = if y $\in R^n$ then $T_y = diag(y_1, y_2 .... y_n)$
Y = y; $|y|$ = e, y $\in R^n$
$A_{yz} = A_c - T_y \Delta T_z$
$A_{ye} = A_c - T_y \Delta$
$A_{yf} = A_c + T_y \Delta$
$D_y = A_c^{-1} T_y \Delta$
$B = D_y |B| + A_c^{-1}$
ρ_o (A) = Eigenvalues of A
$D_{yz} = A_c^{-1} T_y \Delta T_z$

1.2 Arithmetic Operations on the Set of Intervals

We will the denote intervals and their endpoints by capital letters. The left and right endpoints of an interval X will be denoted by $\underline{X}$ and $\overline{X}$, respectively. Thus, X = [$\underline{X}$, $\overline{X}$]

Let " · " denotes any binary operation, X and Y be any two intervals.

X · Y = [min S, max S], where S = { $\underline{X}\,\underline{Y}$, $\underline{X}\,\overline{Y}$, $\overline{X}\,\underline{Y}$, $\overline{X}\,\overline{Y}$ }

Example 1.1. Let X = [-1, 0] and Y = [1, 2]. Then for X ∗ Y, S = { -1, 0, -2 } and X ∗ Y = [-2, 0]

1.3 Interval Matrix

Definition 1.1. An interval matrix A^I is a matrix whose each elements are intervals.

A genral form of an interval matrix is

$$A^I = \begin{pmatrix} a^I_{11} & a^I_{12} & ... & a^I_{1n} \\ a^I_{21} & a^I_{22} & ... & a^I_{2n} \\ ... & ... & ... & ... \\ a^I_{m1} & a^I_{m2} & ... & a_{mn}{}^I \end{pmatrix} = (a^I_{ij})_{1\le i\le m, 1\le j\le n}$$

where each a^I_{ij} = [$\underline{a}_{ij}$,$\overline{a}_{ij}$] (or) A^I = [$\underline{A}$, $\overline{A}$] for some $\underline{A}$, $\overline{A}$ satisfying $\underline{A} \le \overline{A}$

Example 1.2. Take m = n = 2; Let $A^I = \begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [4,7] \end{bmatrix}$: $A^I = [\underline{A}, \overline{A}] = \{A ; \underline{A} \le A \le \overline{A}\}$

Here, $\underline{A} = \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}$, $\overline{A} = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$

Centre: $A_c = \frac{1}{2}[\underline{A} + \overline{A}] = \frac{1}{2} \left[\begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}\right]$

$= \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$

Spread: $\Delta = \frac{1}{2}\ (\underline{A} - \overline{A}) = \frac{1}{2} \left[\begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}\right] =$

$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

Also, any A^I = [$A_c - \Delta$, $A_c + \Delta$], it can be easily shown.

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1.4 Determinant of an Interval Matrix

det A^I = { det A | A ∈ A^I } = [$\min_{\forall A \in A^I}$ det A , $\max_{\forall A \in A^I}$ det A]

Example 1.3. Take, $A^I = \begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [4,7] \end{bmatrix}$; Let S $= \left\{ \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 1 \le a \le 2, 3 \le b \le 4, 5 \le c \le 6, 4 \le d \le 7 \right\}$

= { ad-bc : $1 \le a \le 2$, $3 \le b \le 4$, $5 \le c \le 6$, $4 \le d \le 7$ }

So, det A^I = [min S, max S] = [-20, -1]

2 Regular Interval Matrix

Definition 2.1. A square interval matrix A^I is called regular if each A ∈A^I is nonsingular that is det A ≠ 0.

Example 2.1. S = { ad-bc : $1 \le a \le 2$, $3 \le b \le 4$, $5 \le c \le 6$, $4 \le d \le 7$ } , if $0 \notin S$, then each A is non singular

det A^I = det $\begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [4,7] \end{bmatrix}$ = [-20, -1]

since $0 \notin$ [-20,-1]; A^I will be regular

Theorem 2.1. [6] *Let A^I be regular, then for each A, A_1 ∈A^I, both $A^{-1}A_1$ and $A_1^{-1}A$ are P-matrices.*

Example 2.2. $A = \begin{bmatrix} 1 & 3 \\ 6 & 4 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$;

A^{-1} = - $\frac{1}{14}$ $\begin{bmatrix} 4 & -3 \\ -6 & 1 \end{bmatrix}$

$A^{-1} A_1 = \begin{bmatrix} 1/2 & 0 \\ 5/4 & 1 \end{bmatrix}$; $\Delta_1 = \frac{1}{2} > 0$, $\Delta_2 = \frac{1}{2} > 0$; hence $A^{-1} A_1$ is P-matrix

Similarly, for A_1^{-1} A.

Theorem 2.2. [6] *Let A^I be an nXn interval matrix. Then the following conditions are mutually equivalent:*

(R) A^I is regular .

(A1) $A_{yz}x= y$, $T_z\, x \ge 0$ has a (unique) solution for each $y \in Y$; where, $A_{yz} = A_c - T_y\, \Delta\, T_z$, $Y = \{ y ;|y| = e, y \in R^n \}$.

(A2) $A_{ye}x_1$-$A_{yf}x_2$=y, $x_1 \ge 0$, $x_2 \ge 0$ has a solution for each $y \in Y$, where $A_{ye} = A_c - T_y\, \Delta$, $A_{yf} = A_c + T_y\, \Delta$.

(A3) $B=D_y\, |B| + A_c^{-1}$ has a unique matrix solution for each $y \in Y$, where $D_y = A_c^{-1}\, T_y\, \Delta$.

(B1) $A_{ye}^{-1}A_{yf}$ is a P-Matrix for each y∈Y

(B2) $A_{ye}^{-1}A_{yf}\, x>0$, $x>0$ has a solution for each y∈Y .

(B3) $A_{ye}^{-1}x>0$ and $A_{yf}^{-1}x>0$ has a solution for each y∈Y.

(B4) If A^I is regular matrix.Then $|D_y x| < x$ has a solution for each y∈Y.

(C1) (det A_{yz})(det $A_{y'z'}$) >0 for each y, y′, z, z′ ∈ Y.

(C2) (det A_{yz})(det $A_{y'z}$)>0 for each y,y′,z∈Y such that y and y′ differ in just one entry

(C3) $\rho_o(D_{yz})$<1 for each y , z ∈ Y, where $D_{yz} = A_c^{-1} T_y\, \Delta\, T_z$

(C4) $(A_c\, A_{yz}^{-1})_{ii} > 1/2$; for each y, z∈Y; i∈1,2,...,n.

(C5) $(A_c A^{-1})_{ii} > 1/2$ for each $A \in A^I$, i ∈ i,2,...,n

Example 2.3. Take n = 2 then

$Y = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

(R)→(A1)

$z = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $T_z = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; $T_y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

$A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$; $\Delta = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

$A_{yz} = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5/2 \\ 5 & 5/2 \end{bmatrix}$

$A_{yz}x = y \Rightarrow \begin{bmatrix} 1 & 5/2 \\ 5 & 5/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

$x = \begin{bmatrix} 0 \\ 2/5 \end{bmatrix}$; $T_z x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/5 \end{bmatrix} \ge 0$

(R)→(A2)

$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Ty = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$A_{ye} = A_c - T_y\Delta = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix}$

$A_{yf} = A_c + T_y\Delta = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix}$

$A_{ye}x_1$ - $A_{yf}x_2 = y$; $x_1 \ge 0$, $x_2 \ge 0$

$\begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x'_1 \\ x''_1 \end{bmatrix}$- $\begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x'_2 \\ x''_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; By solving $x_1 = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

(R)→(A3)

$A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$; $A_c^{-1} = \begin{bmatrix} -11/44 & 7/44 \\ 11/44 & -3/44 \end{bmatrix}$

take $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $\Delta = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

$D_y = A_c^{-1}\, T_y\Delta = \begin{bmatrix} -11/44 & 7/44 \\ 11/44 & -3/44 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} =$ $\begin{bmatrix} -18/88 & -32/88 \\ 14/88 & 20/88 \end{bmatrix}$ take $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|B| = \begin{bmatrix} |a| & |b| \\ |c| & |d| \end{bmatrix}$

$B = D_y|B| + A_c^{-1} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$

$\begin{bmatrix} -18/88 & -32/88 \\ 14/88 & 20/88 \end{bmatrix} \begin{bmatrix} |a| & |b| \\ |c| & |d| \end{bmatrix}$
$+ \begin{bmatrix} -11/44 & 7/44 \\ 11/44 & -3/44 \end{bmatrix}$

By solving this we get,
$B = \begin{bmatrix} -0,2874 & 0.1494 \\ 0.2644 & -0.0575 \end{bmatrix}$

(R)→(B1)

$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $A_{ye} = \begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix}$; $A_{yf} = \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix}$

$A_{ye}^{-1} A_{yf} = -\frac{1}{11} \begin{bmatrix} 7 & -3 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix}$
$= \begin{bmatrix} 1/11 & -16/11 \\ 7/11 & 20/11 \end{bmatrix}$

$\Delta_1 = 1/11 \geq 0$ and $\Delta_2 = 132/121 \geq 0$.
So $A_{ye}^{-1} A_{yf}$ is a P-Matrix

(R)→(B2)

take $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $A_{ye}^{-1} A_{yf} = \begin{bmatrix} 1/11 & -16/11 \\ 7/11 & 20/11 \end{bmatrix}$

$A_{ye}^{-1} A_{yf} x > 0 \Rightarrow \begin{bmatrix} 1/11 & -16/11 \\ 7/11 & 20/11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0; x = \begin{bmatrix} 17 \\ 1 \end{bmatrix} > 0$

(R)→(B3)

take $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $A_{ye}^{-1} = -\frac{1}{11} \begin{bmatrix} 7 & -3 \\ -6 & 1 \end{bmatrix}$
$= \begin{bmatrix} -7/11 & 3/11 \\ 6/11 & -1/11 \end{bmatrix}$

$A_{ye}^{-1} x = \begin{bmatrix} -7/11 & 3/11 \\ 6/11 & -1/11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$; Solving this we get $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$A_{yf} = \begin{bmatrix} 2 & 4 \\ 5 & 4 \end{bmatrix}$; $A_{yf}^{-1} = \begin{bmatrix} -4 & 4 \\ 5 & -2 \end{bmatrix}$

$A_{yf}^{-1}\ x > 0 \Rightarrow \begin{bmatrix} -4 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$; So the value of x is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(R)→(B4) take $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; from previous examples we know $D_y = \begin{bmatrix} -18/88 & -32/88 \\ 14/88 & 20/88 \end{bmatrix}$

$D_y x = \begin{bmatrix} -18/88 & -32/88 \\ 14/88 & 20/88 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$= \begin{bmatrix} -18x_1/88 & -32x_2/88 \\ 14x_1/88 & 20x_2/88 \end{bmatrix}$; $|D_y x| <$ x means

$|-18x_1/88 - 32x_2/88| < x_1$ and $|14x_1/88 + 20x_2/88| < x_2$

solving this we get $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(R) →(C1)

$A_{yz} = A_c - T_y \Delta T_z$; take $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $z' = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$T_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $T_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_{y'} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $T_{z'} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; $A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$

$T_y \Delta\ T_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -3/2 \end{bmatrix}$

$A_{yz} = A_c - T_y\ \Delta\ T_z = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix} - \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -3/2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix}$

$T_{y'}\ \Delta\ T_{z'} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -3/2 \end{bmatrix}$

$A_{y'z'} = A_c - T_{y'}\ \Delta\ T_{z'} = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix} - \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -3/2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 5 & 7 \end{bmatrix}$

det A_{yz} = 7-18 = -11
det $A_{y'z'}$ = 7-20 = -13
(det A_{yz}) (det $A_{y'z'}$) = (-11) ·(-13) = 143 > 0

(R)→ (C2)

take $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $y' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $A_{yz} = \begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix}$

$A_{y'z} = A_c - T_{y'}\ \Delta\ T_z = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$

det A_{yz} = 7 - 18 = -11
det $A_{y'z}$ = 14 - 24 = -10
(det A_{yz})(det $A_{y'z}$) = (-11) · (-10) = 110 > 0

(R)→(C3)

take $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $T_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $T_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\Delta = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

$T_y\ \Delta\ T_z = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -3/2 \end{bmatrix}$; $D_{yz} = A_c^{-1} T_y\ \Delta\ T_z$
$= \begin{bmatrix} -11/44 & 7/44 \\ 11/44 & -3/44 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -3/2 \end{bmatrix}$

$= \begin{bmatrix} -18/88 & -32/88 \\ 14/88 & 20/88 \end{bmatrix}$

So, after calculating the eigen value of D_{yz} = 0.117 and -0.094, where both are less than 1.

(R)→(C4)

take y = $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and z = $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$;

$T_y = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $T_z = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$; $\Delta = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

$A_{yz} = A_c - T_y\ \Delta\ T_z = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$

$- \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 5 & 7 \end{bmatrix}$

$A_{yz}{}^{-1} = \begin{bmatrix} -7/13 & 4/13 \\ 5/13 & -1/13 \end{bmatrix}$

$A_c A_{yz}{}^{-1} = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix} \begin{bmatrix} -7/13 & 4/13 \\ 5/13 & -1/13 \end{bmatrix}$

$= \begin{bmatrix} 14/26 & 5/26 \\ -22/26 & 33/26 \end{bmatrix}$

$(Ac\ A_{yz}{}^{-1})_{11} = 14/26 = 0.538 > 1/2$

$(Ac\ A_{yz}{}^{-1})_{22} = 33/26 = 1.269 > 1/2$

(R)→(C5)

$A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix}$; $A^I = \begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [4,7] \end{bmatrix}$

take $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} -7/8 & 3/8 \\ 5/8 & -1/8 \end{bmatrix}$

$A_c A^{-1} = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 11/2 \end{bmatrix} \begin{bmatrix} -7/8 & 3/8 \\ 5/8 & -1/8 \end{bmatrix}$

$= \begin{bmatrix} 14/16 & 2/16 \\ -22/16 & 22/16 \end{bmatrix}$

$(A_c A^{-1})_{11} = 14/16 > 1/2$

$(A_c A^{-1})_{22} = 22/16 > 1/2$

3 Algorithm for checking Singularity of an Interval Matrix

Definition 3.1. An intertval matrix A^I is called singular interval matrix if it contains a singular matrix that is for any $A \in A^I$, det A = 0.

Example 3.1. det A^I = det $\begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [7,8] \end{bmatrix}$ = [-17 , 1]

since $0 \in$ [-17 , 1]; so A^I contains atleast one singular matrix.

Algorithm: [6]

Step 0 :: Select a matrix A such that $|A - A_c| = \Delta$ [recommended: $A_{ij} = \underline{A}_{ij}$ if $(A_c{}^{-1})_{ji} \geq 0$ and $A_{ij} = \overline{A}_{ij}$ otherwise]

Step 1:: Compute A^{-1}.

Step 2::If $K_i = \phi$ for each i, terminate the algorithm fails. where K_i={ j; such that $(A_c\text{-}A)_{ij} A^{-1}{}_{ji} < 0$}.

Step 3:: Otherwise find k such that $K_k \neq \phi$ and ψ_k= min { ψ_j; $K_j \neq \phi$ } where

$$\psi_i = \sum_{j \in K_i} (A_c - A)_{ij} A_{ji}{}^{-1}$$

Step 4:: If $\psi_k \leq$ -1/2 terminate. A^I is singular.

Step 5:: Otherwise set $A_{kj} = (2A_c\text{-}A)_{kj}$ for each $j \in K_k$ and go to step 1.

Example 3.2. We have taken an interval matrix $A^I = \begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [7,8] \end{bmatrix}$; Now we will verify this.

$A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 15/2 \end{bmatrix}$

Step 0 : $\Delta = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$; $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

Since, $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 15/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$;

$|A - A_c| = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \Delta$

Step 1 : $A^{-1} = \begin{bmatrix} -1 & 1/2 \\ 3/2 & -1/4 \end{bmatrix}$

Step 2 : A_c - A = $\begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 15/2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}$

$(A_c - A)_{11}\ A_{11}{}^{-1}$ = (-1/2) (-1) = 1/2 > 0 and $(A_c - A)_{12}\ A_{21}{}^{-1}$ =(-1/2)(3/2) = -3/4<0

$K_1 = 2$

$(A_c - A)_{21}\ A_{12}{}^{-1}$ = (-1/2) (1/2) = -1/4 ¡ 0 and $(A_c - A)_{22}\ A_{22}{}^{-1}$= (-1/2)(-1/4) =1/8>0

$K_2 = 1$

Step 3 : $K_1 = 2 \neq \phi$ and $K_2 = 1 \neq \phi$

$\psi_1 = \sum_{j \in K_1} (A_c - A)_{1j}\ A_{j1}{}^{-1} = \sum_{j=2} (A_c - A)_{12} A_{21}{}^{-1} = -3/4$

$\psi_2 = \sum_{j \in K_2} (A_c - A)_{2j} A_{j2}{}^{-1} = \sum_{j=1} (A_c - A)_{21} A_{12}{}^{-1} = -1/4$

ψ_k = min { -3/4 , -1/4 } = -3/4 = ψ_1

Step 4 : ψ_1 = -3/4 $\leq$ -1/2

So according to the algorithm given A^I is singular.

4 Theoretical Result on Singular Interval Matrix

Theorem 4.1. [7] *An interval matrix A^I is singular if and only if it satisfies $|A_c x| \leq \Delta\ |x|$; for some non zero vector x*

Example 4.1. take $A^I = \begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [7,8] \end{bmatrix}$;

$A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 15/2 \end{bmatrix}$; $\Delta = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$|A_c x| \leq \Delta\ |x| \Rightarrow \left| \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 15/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|$
$\leq \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \left| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|$

solving this we get x = $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

Conversely, take $A_c = \begin{bmatrix} 3/2 & 7/2 \\ 11/2 & 15/2 \end{bmatrix}$; $\Delta = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and x = $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ which satisfies the above inequality.

So, from A_c and Δ we can get $A^I = [A_c - \Delta, A_c + \Delta] = \begin{bmatrix} [1,2] & [3,4] \\ [5,6] & [7,8] \end{bmatrix}$, which is a singular matrix.

5 Eigen Values of an Interval Matrix

Definition 5.1. Let A ∈ $M_{nxn}(R)$. A nonzero vector v ∈ R^n is called an eigen vector of A if Av = λv for some scalar λ. The scalar λ is called the eigenvalue of A.

Example 5.1. Let A = $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$; we will get eigenvalues and eigen vectors after solving Av = λv.

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ v = λ v; solving this we get eigen values are 3, -1 and corrosponding eigen vectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively.

Theorem 5.1. [7] $\lambda \in$ L if and only if the interval matrix $[(A_c - \lambda I) - \Delta, (A_c - \lambda I) + \Delta]$ is singular matrix. *where* $L = \{ \lambda \in R;\ Ax = \lambda x$ *for some,* $A \in A^I,\ x \neq 0 \}$

Example 5.2. Take $A^I = \begin{bmatrix} [1,3] & [3,5] \\ [5,7] & [7,9] \end{bmatrix}$; $A_c = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$; $\Delta = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Take any A ∈ A^I; A = $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$; say eigen values of A are λ_1 and λ_2

λ_1 = 12.0827 and λ_2 = -0.08276

$(A_c - \lambda_1 I) - \Delta = (A_c - \lambda_1 I) + \Delta = \left(\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 12.082 & 0 \\ 0 & 12.082 \end{bmatrix} \right) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -9.082 & 5 \\ 7 & -3.082 \end{bmatrix}$

Now the new interval matrix will be = $[(A_c - \lambda_1 I) - \Delta, (A_c - \lambda_1 I) + \Delta]$

$A^{I\ \prime} = \begin{bmatrix} [-11.082, -9.082] & [3,5] \\ [5,7] & [-5.082, -3.082] \end{bmatrix}$; $A_c{}' = \begin{bmatrix} -10.082 & 4 \\ 6 & -4.082 \end{bmatrix}$ and $\Delta' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Since we know A^I is singular if and only if it satisfies $|A_c x| \leq \Delta\ |x|$; for some non zero vector x

$|A_c{}'\ x| \leq \Delta'\ |x| \Rightarrow \left| \begin{bmatrix} -10.082 & 4 \\ 6 & -4.082 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|$
$\leq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \left| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right|$

Solving this we get x = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; So $A^{I\ \prime}$ is singular.

Conversely, take $A_c = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, $\Delta = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and λ = 12.0827

Then $[(A_c - \lambda I) - \Delta, (A_c - \lambda I) + \Delta] = \begin{bmatrix} [-11.082, -9.082] & [3,5] \\ [5,7] & [-5.082, -3.082] \end{bmatrix}$ is singular.

for λ = 12.0827 ∃ A = $\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix} \in \begin{bmatrix} [1,3] & [3,5] \\ [5,7] & [7,9] \end{bmatrix}$

Theorem 5.2. [7] *Let* $\lambda \in L$*, then* $\exists\ x \neq 0,\ p \neq 0$ *and* $y, z \in Y$ *and* $d \in [0, 1]$ *such that,*

1) $(A_c - dT_y\ \Delta\ T_z)\ x = \lambda\ x$

2) $(A_c - dT_y\ \Delta\ T_z)^t\ p = \lambda\ p$

3) $T_z x \geq 0$

4) $T_z p \geq 0$

Note : When A^I is a 2x2 interval matrix then

$$Y = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

6 Present Contribution for determining Eigen Value

6.1 Matlab Code for 2x2 matrix

```
    function    X=centre(a,b,c,d,e,f,g,h)
i=(a+b)/2;
j=(c+d)/2;
k=(e+f)/2;
l=(g+h)/2;
X=[i,j;k,l];
end
function Y=spread(a,b,c,d,e,f,g,h)
i=(b-a)/2;
j=(d-c)/2;
k=(f-e)/2;
l=(h-g)/2;
Y=[i,j;k,l];
end
    function X=VECTOR(a,b,c,d,e,f,g,h)
C=centre(a,b,c,d,e,f,g,h);
D=spread(a,b,c,d,e,f,g,h);
n=1000;
d= 0.0;
for i=1:n
d=d+0.001;
A=C-d*(diag([1,-1])*D*diag([-1,1]));
B=A';
[V,W]= eig(A)
z=diag([-1,1])* V;
z1=z*[1;0]
z2=z*[0;1]
q=all(all(z1 >= 0))
p=all(all(z2 >= 0))
[V,W]=eig(B)
Z=diag([-1,1])*V;
Z1=Z*[1;0]
Z2=Z*[0;1]
Q=all(all(Z1 >= 0))
P=all(all(Z2 >= 0))
if(q==1 || p==1 || P==1 || Q==1)
disp ('done for d ')
d
break
end
end
end
```

Table 1

Matrix	y	z	Eigen Value	Eigen Vectors (x and p)	d
$\begin{bmatrix} [1,3] & [4,6] \\ [6,8] & [6,10] \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	-1.6311	$\begin{bmatrix} -0.8090 \\ 0.5878 \end{bmatrix}$; $\begin{bmatrix} -0.8876 \\ 0.4606 \end{bmatrix}$	0.001
$\begin{bmatrix} [1,3] & [5,7] \\ [9,11] & [14,15] \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	-1.7013	$\begin{bmatrix} -0.8510 \\ 0.5252 \end{bmatrix}$; $\begin{bmatrix} -0.9378 \\ 0.3472 \end{bmatrix}$	0.001
$\begin{bmatrix} [10,12] & [14,16] \\ [18,20] & [22,24] \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	-0.9145	$\begin{bmatrix} -0.7830 \\ 0.6220 \end{bmatrix}$; $\begin{bmatrix} -0.8472 \\ 0.5313 \end{bmatrix}$	0.001

7 Conclusion

In this paper, we have calculated the eigen value of an interval matrix using singularity property and developed matlab code.

Acknowledgement : I am heartily thankful to **Prof. Geetanjali Panda**, IIT Kharagpur for her constant encouragement throughout this work. Her valuable advices and guidance were a constant source of inspiration for me.

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